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Potential applicability of Mesh - Free Method Using Penalty Approach

MUTHU VIJAY P., NANDAKISHORE, BHAVANA PATEL, BABUNARAYAN K. S.
Department of Civil Engineering, NITK, Surathkal, Karnataka, India
Email: vjyprbhr@gmail.com, nandakishorecv25@gmail.com

Abstract: The accuracy of a FEM solution depends upon how well the structure is discretized, whereas Mesh free method works on the nodes and their domains to arrive at the solution without discretization of the structure. An attempt has been made to develop a purely mesh less method. An example has been illustrated to demonstrate the potential application of the same. No background mesh is employed in this process even for nodal integration. Complexity involved with the numerical integration using Gaussian quadrature is completely ruled out. MLS procedure is deployed to arrive at the shape functions. Penalty approach is used to enforce the boundary condition. An algorithm based on MATLAB coding is developed to obtain displacement profile along the length of the 2-D cantilever loaded with a point load at free end. The result obtained shows good agreement with the analytical solution and FEM solution.

Keywords: *Mesh-free method, Moving Least Square, EFG, Penalty method, nodal direct integration, purely meshless method, shape function, MATLAB coding*

Introduction:

All the physical phenomena encountered in engineering are modelled by differential equations. To solve the Differential Equations (DE), two major approaches are followed. They are:

i. Analytical and
ii. Numerical.

Analytical approach leads to closed-form solutions and is effective in case of simple geometry, boundary conditions, loadings and material properties. For most of practical problems where it is not possible to get exact analytical solution, Numerical solutions are being called for. Strong form method discretizes and solves the governing DE directly. In weak form method instead of solving DE of the underlying problem directly, an integral function that governs the same physical phenomena is solved. Weakened weak form reduces the order of DE with respect to weak form and then solves it. The various numerical methods available are: FDM, BEM, FVM, FEM, X-FEM, Mesh Free Method.

This work is a preliminary attempt to understand and find out the applicability of purely meshless method for the problem in hand.

Mesh-Free Method:

1 Why Mesh-Free Methods?

Though FEM is a robust and thoroughly developed method, and widely used in engineering fields due to its versatility for complex geometry and flexibility for many types of linear and non-linear problems and also most practical engineering problems are currently solved using well developed FEM packages, it has lots of drawbacks. The following limitations of FEM are becoming increasingly evident:

- FEM rely on meshes or elements that are connected together by nodes in a properly predefined manner and creation of a quality mesh requires more man power than the computational efforts.
- The compatible FEM model is usually overly stiff and only provides a lower bound solution.
- Limitations in the analyses of some problems: It is difficult to simulate crack growth and breakage of materials in FEM. Under large deformations, due to element distortion FEM results will not be accurate.
- FDM works well only for regularly distributed nodes. Studies are still going on to develop methods using irregular grids.
- It is difficult in using X-FEM when we are dealing with complete fragmentation of the structure.
- And also we can’t increase the order of polynomial in FEM (unless re-meshing is done) or in X-FEM to increase the accuracy in the result since it is based on element. Whereas in MFM we can increase the order of polynomial since it is based on nodes.

The root of these problems is the use of elements or mesh in the formulation stage. The idea of getting rid of the elements and meshes in the process of numerical treatments has naturally evolved, and the concepts of mesh free or mesh less methods have been shaped up. To overcome all this shortcomings a new method was proposed which even don’t require mesh at all, which is named as element-free / mesh-less / mesh-free method.

2 Definitions:

It is a method used to establish a system of algebraic equations for the whole problem domain without the use of a predefined mesh or uses easily generable meshes in a much more flexible or freer manner. It uses a set of field nodes scattered within the problem domain as well on the boundaries of domain to represent the problem and its boundaries.

Mesh-Free techniques can be conveniently used in the problems of solid mechanics like stress concentration and crack growths, wherein more number of nodes have to be introduced near the regions of stress concentrations and at the cracks. It is also well suited for problems on large deformations and discontinuities. Ideally Mesh-Free method should not use any mesh but the Mesh-Free methods developed so far are not entirely mesh-free but it requires background cells for the integration of system matrices.

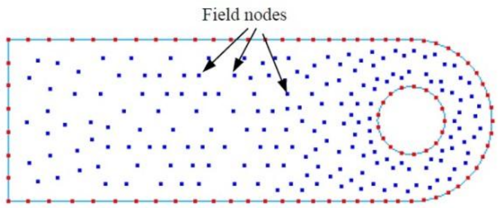

Figure 1: Distribution of nodes in domain by MFM (Liu.R.G, 2005)

There are a number of mesh-free methods that use local nodes for field variable approximation, such as

- Smooth Particle Hydrodynamics (SPH),
- Diffuse Element Method (DEM),
- Element Free Galerkin method(EFG),
- Mesh-less Local Petrov–Galerkin (MLPG) method,
- Reproducing Kernel Particle Method(RKPM),
- Point Interpolation Method (PIM),
- Finite Point Method(FPM),
- FDM with arbitrary irregular grid,
- Local Point Collocation methods.

Apart from these methods, the developments of "merging" or "fusing" different methods (even with FDM, FEM) are very important in inventing more effective computational methods for more complicated engineering problems.

3 Procedure of MFM:

The procedure of Mesh Free method is briefed step by step in the following flow chart,

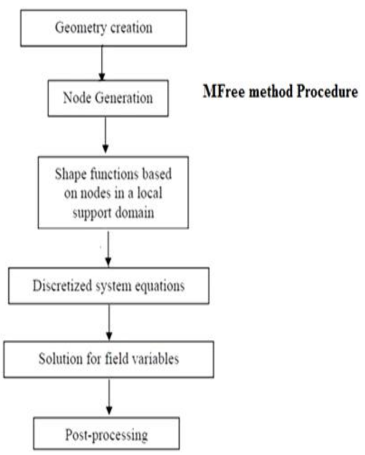

Figure 2: Flow chart comparison of procedures of FEM & MFM (Liu.R.G, 2005).

4 Moving Least Squares Shape Functions

The shape function is the function which interpolates the solution between the discrete values obtained at the mesh nodes. In simple words, it tells about the variation of deflection between the nodes. It doesn't satisfy the Kronecker's delta property (as in FEM) in Mesh free method.

Kronecker's delta property (preferable condition):

$$\delta_{ij} = 0 \text{ if } i \neq j,$$
$$\delta_{ij} = 1 \text{ if } i = j.$$

Partition of unity (compulsory condition):
Total summation of all shape functions "φ" at any node is always equal to 1

$$\sum_{i=1}^{n} \phi_i(x) = 1$$

The value of shape function will vanish outside the support domain.

The shape function construction is formulated in this work by the use of MLS technique.

The moving least squares (MLS) approximation was devised by mathematicians in data fitting and surface construction (Lancaster and Salkausdas 1981; Cleveland 1993). It can be categorized as a method of series representation of functions. The MLS approximation is now widely used in Mesh Free methods for constructing Mesh Free shape functions.

Formulation:
Shape function or interpolation of field variables decides the accuracy of the results. u(x, y) is the function of field variable defined in the domain. If the approximation of u(x, y) at a point is given as uh(x, y), then the MLS approximation can written as,

$$u^h(x) = \sum_{j}^{m} p_j(x) a_j(x) = p^T(x)\, a(x)$$

Where p(x,y) is the basis function of the spatial coordinates, and m is the number of the basis functions. The basis function p(x,y) is often built using monomials from the Pascal triangle to ensure minimum completeness.

a(x) is the vector of coefficients given by
$a^T(x) = \{ a_0(x)\ a_1(x) \ldots\ a_m(x)\}$ which are functions of x. The coefficients a can be obtained by minimizing the following weighted residual function.

$$J = \sum_{I}^{n} W\ (x - x_I)[u^h(x, x_I) - u(x_I)]^2$$
$$J = \sum_{I}^{n} W\ (x - x_I)[p^T(x_I)a(x) - u_I]^2$$

where,
$W\ (x - x_I)$ is a weight function, chosen so that to have the following properties.
$W\ (x - x_I) > 0$ within the support domain.
$W\ (x - x_I) = 0$ outside the support domain.
$W\ (x - x_I)$ Monotonically decreases from point of interest x.
$W\ (x - x_I)$ is sufficient, smooth , especially on the boundary.

Weight functions play an important role in mesh-free methods. They should be constructed according to the reproducibility requirement. Most mesh-free weight functions are bell shaped.

The various weight functions are,

- The cubic spline function
- The quartic spline function
- The exponential function

The major requirement for an efficient weight function are listed below,

i. Unity condition states that the weight function is one at the centre of the support domain (r_i = 0)

$W_{(ri = 0)} = 1$

ii. Compact support condition states that 1st and 2nd derivatives of the weight function are all zero at the boundary of the support domain (r_i = 1). This compact support condition leads to the following set of equations.

$W_{(ri\,=\,1)} = 0$

$$(\partial W/\partial r)_{(ri\,=\,1)} = 0$$

$$(\partial^2 W/\partial r^2)_{(ri\,=\,1)} = 0$$

iii. The condition of symmetry states that the 1st derivative of the weight function is zero at the centre of the support domain (r_i = 0). The condition of symmetry gives the following equation.

$$(\partial W/\partial r)_{(ri\,=\,0)} = 0$$

Based on these criteria the exponential weight function is chosen out of the above mentioned weight functions.

The exponential weight function is expressed as,

$$W_i(x) = \begin{cases} e^{(r_i/\alpha)^2} & r_i \leq 1 \\ 0 & r_i > 1 \end{cases}$$

where,

α=0.3 (based on literatures)

$r_i = d_i / r_w$

d_i - distance between point of interest and the node considered.

r_w - size of the support domain.

n - number of nodes in the support domain of x for which the weight function $W\ (x - x_I) \neq 0$ and u_i is the nodal parameter of u at $x=x_i$ is a functional, a weighted residual, that is constructed using the approximated values and the nodal parameters of the unknown field function.

The stationary of J with respect to a(x) gives

$$\frac{\partial J}{\partial a} = 0$$

which leads to,

$\Phi^T(x) = \{ \Phi_1(x), \Phi_2(x), \ldots, \Phi_n(x)\}_{(1\,*n)}$

$$\mathbf{\Phi^T(x) = p^T (x) * A^{-1} (x) * B (x)}$$

where,

Φ(x) - shape function matrix

$A(x) = \sum_{i=1}^{n} W_i(x) p(x_i) p^T(x_i)$

$B(x) = \sum_{i=1}^{n} W_i(x) p(x_i)$

$p(x_i) = [\ 1; x_i; y_i; x_i y_i]$

5 Enforcement of Boundary condition

Imposing essential boundary conditions is a key issue in mesh-free methods. The Mesh-free interpolation does not verify the Kronecker delta property and, therefore, the imposition of prescribed values is not as straightforward as in the finite element method. To enforce the boundary condition at the boundary nodes in Mesh-Free method, two methods are followed widely. They are

i. Lagrange Multiplier method and
ii. Penalty method.

In the present study, penalty method is used to enforce essential boundary conditions. The salient features of this method are briefed bellow.

5.1 Penalty method

The penalty method is more convenient method for enforcing the essential boundary conditions, due to the advantage that there are only two changes of matrices and the algorithm is very simple. The procedure of penalty method is explained below.

Based on the practice in FEM, the penalty coefficient 'α' can be determined as,

$$\alpha = 10^4 \sim 10^8\ (K)_{max}$$

where,

α - penalty coefficient.

$(K)_{max}$ - maximum diagonal element of the global stiffness matrix.

Using α, nodal penalty stiffness matrix (K^{α}_{IJ}) and nodal penalty force vector (F^{α}_{J}) are formulated as shown below,

$K^{\alpha}_{IJ} = \int \varphi_I^T \alpha\ \varphi_J d\Gamma$

$F^{\alpha}_{J} = \int \varphi_I^T \alpha\ u_\Gamma d\Gamma$

where,

K^{α}_{IJ} – nodal penalty stiffness matrix,

φ_I , φ_J – Shape function matrices for boundary nodes considered.

F^{α}_{J} - nodal penalty force vector.

The integrations in the penalty stiffness matrix and the penalty force vector is done and they are assembled to form the global penalty stiffness matrix (K^{α})and global penalty force vector (F^{α}). The integration is performed along the essential boundary (Γ), and hence matrix K^{α} will have entries only for the nodes near the essential boundaries Γ_u.

Substituting the foregoing expression for all the displacement components of **u** in the general Equation, yields the following global discretized system equations of the EFG method.

$$\mathbf{[K + K^{\alpha}]\ U = F + F^{\alpha}}$$

where,

U - vector of nodal parameters of displacements for all nodes in the entire problem domain,

K - global stiffness matrix assembled using the nodal stiffness matrices, and

F - global external force vector assembled using the nodal force vectors,

As the boundary conditions are enforced, The Galerkin procedure makes the stiffness matrices K and K^{α} symmetric. If the problem domain is sufficiently supported without rigid body movement, [K + K^{α}] will be positive definite; now it is possible to invert the stiffness matrix as it becomes non-singular positive definite matrix. A standard linear algebra equation solver can be used to solve Equation for the nodal displacement parameters.

The advantage of using the penalty method is that the dimension, symmetry and positive definite properties of the stiffness matrix are achieved, as long as the penalty factors chosen are positive. In addition, the symmetry and the bandness of the system matrix are preserved.

However, the penalty method has the following shortcomings.

- The penalty method can only approximately satisfy the essential boundary conditions. depending on the magnitude of the penalty coefficients. (larger the penalty coefficients, the more accurate the enforcement of the essential boundary conditions)
- It is difficult to choose a set of penalty factors that are universally applicable for all kinds of problems. One hopes to use large possible penalty factors, but too large penalty factors often give numerical problems, as we experienced in the imposition of multi-point boundary condition in the finite element methods. Trials may be needed to choose a proper penalty factor.
- The results obtained are generally less accurate than those obtained from the method of Lagrange multipliers method.

Despite these disadvantages, the penalty method is widely used because of its low computational cost and quick convergence of solution than Lagrange multiplier.

Literature Review:

- On comparing with FEM, even though Mesh-free Method has some disadvantage like computationally expensive and difficulty in introducing the boundary conditions, it gives more accurate, reliable and stable results than FEM even for typical geometry like corrugated plates and for elasto-plastic analysis of plates.
- Mesh-free is more effective in high stress gradient problems like crack initiation and growth problems and its accurate determination of SIF compared to other methods is better even when the nodes are distributed unevenly.
- EFG method can be made as a truly mesh-less method by the use of Nodal direct integration scheme. Though it helps to avoid shear locking and very accurate, it has a disadvantage of very high computational cost and problem of mathematical instability.
- Coupling of MFM and FEM has the problem in the interface layer of cracks where the shape functions generated by both EFGM and FEM must be compatible.
- There is still ambiguity in deciding the size of influence domain of the node.
- There is a huge complexity involved in usage of Gaussian quadrature technique in numerical integration and also some instability issues involved in direct nodal integration.
- Moving Least Squares approximation, requires only nodal data and no element connectivity, and therefore is more flexible than the conventional Finite element method. But the resulting shape function results doesn't possess Kronecker Delta property
- There is a huge problem in enforcing the boundary condition with Mesh-free method since its shape function lacks Kronecker Delta property.
- Enforcement of boundary condition through Lagrangean multiplier and Penalty is used widely today, but there are both merits and de-merits involved with both these methods.
- Purely mesh less methods like SPH and LPC are not being used in practice because of the stability issues which are need to be addressed.

Problem Definition:

A 2D cantilever, ABCD, of dimensions 2500mm * 500mm * 1mm is considered. AB is fixed and a load of 500 N is applied at C. The material properties for the 2D cantilever are listed below:

Table 4.1: *Material Properties of the 2D cantilever.*

Properties	**Value**
Grade of Concrete	M25
Modulus of Elasticity	25000 N/mm^2
Poisson's Ratio	0.15
Coefficient of thermal expansion	12×10^{-6}/K
Modulus of Rigidity	10869.5 N/mm^2

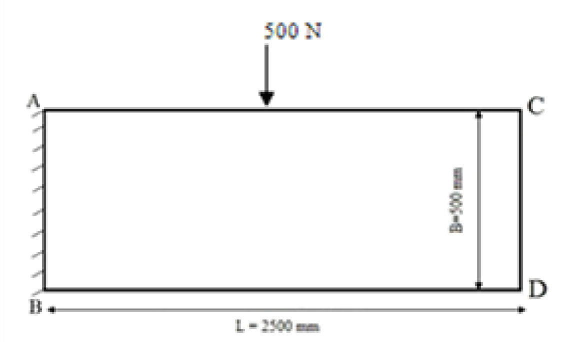

Figure 3: Geometrical conFigureuration of 2D Cantilever

MATLAB coding was developed for triangulated distribution of nodes such that the support domain of point of interest should cover the entire domain with minimum overlap.

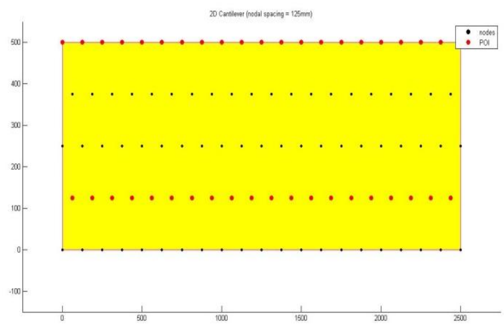

Figure 4: Distribution of nodes on 2D cantilever with nodal spacing of 250mm

The formulation of MATLAB coding for Mesh-Free method, FEM for the analysis of the 2D cantilever problem is given in Appendix in the form of flow chart.

Results and Conclusions:

1 Results:

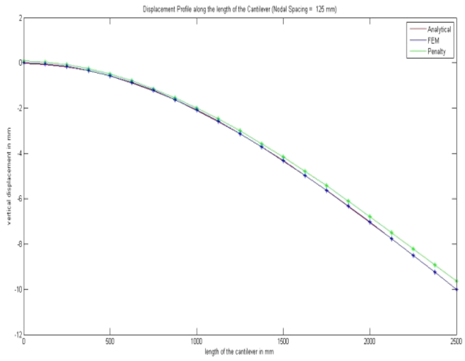

Figure 5: Comparison of displacement profile of 2D cantilever with FEM and Analytical solutions (nodal spacing = 250mm)

A Parametric study was conducted on different nodal combination by varying the spacing of nodes. The nodes are distributed in triangulated fashion and the radius of support domain is fixed as 1.5 times the spacing for all the exercise. The variation of the displacement along the length of the cantilever is captured and compared with analytical and FEM solutions.

Exercise 1: Nodal spacing = 250 mm (Total number of nodes = 32)

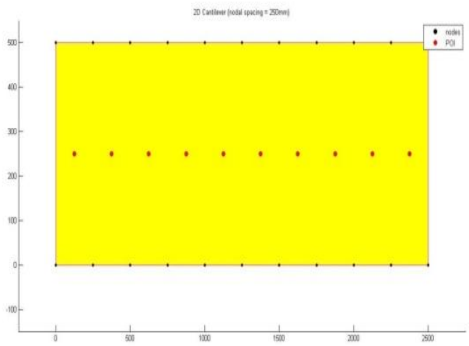

Figure 6: Distribution of nodes on 2D cantilever with nodal spacing of 125mm

The result obtained from EFG method using Penalty approach is compared with FEM and Analytical solutions in a graphical format as shown in the Figure

Since there are very less number of nodes (32), the solution is not converging to the actual solution properly, and the final displacement at free end is 9.5mm.

Exercise 2: Nodal spacing = 125 mm (Total number of nodes = 103)

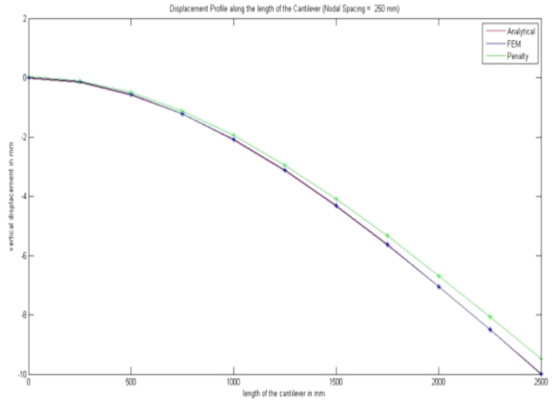

Figure 7: Comparison of displacement profile of 2D cantilever with FEM and Analytical solutions (nodal spacing = 125mm)

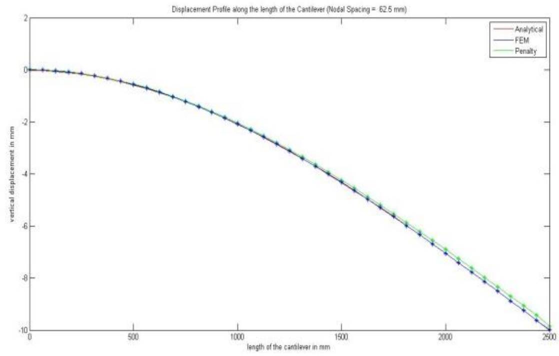

Figure 8: Distribution of nodes on 2D cantilever with nodal spacing of 100mm

The nodal spacing is reduced by 50% so as to have more number of nodes in the domain and trying to capture the difference

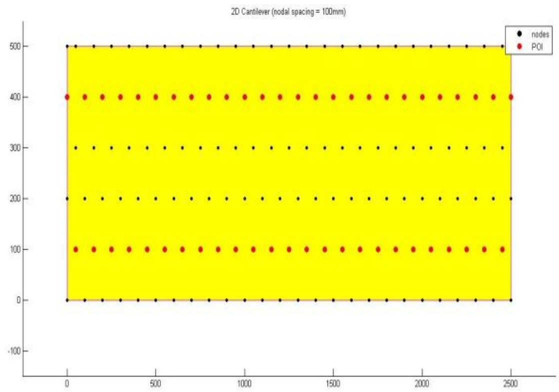

Figure 9: Comparison of displacement profile of 2D cantilever with FEM and Analytical solutions (nodal spacing = 100mm)

Since the nodal spacing is reduced by half, the error is reduced from 5% to 3.5%

Exercise 3: Nodal spacing = 100 mm (Total number of nodes = 155)

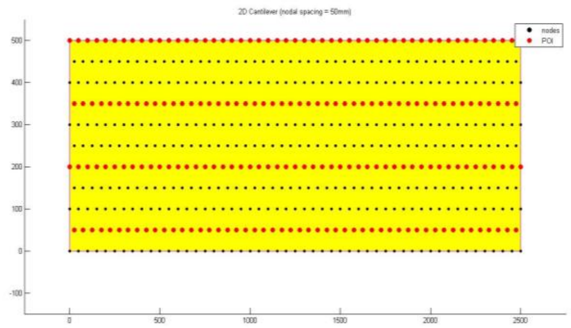

Figure 10: Distribution of nodes on 2D cantilever with nodal spacing of 62.5mm

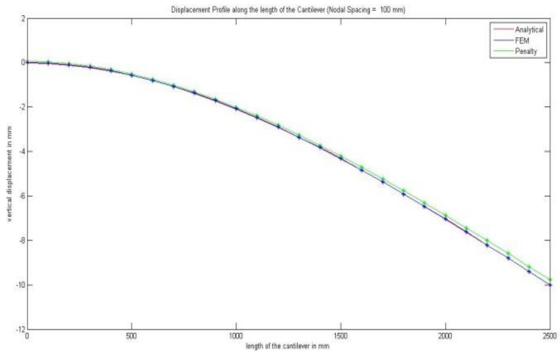

Figure 11: *Comparison of displacement profile of 2D cantilever with FEM and Analytical solutions (nodal spacing = 62.5mm)*

As the nodal spacing is further reduced, the error in the result also starts to converge to the actual, the error being 2.3%.

Exercise 4: Nodal spacing = 62.5 mm (Total number of nodes = 365)

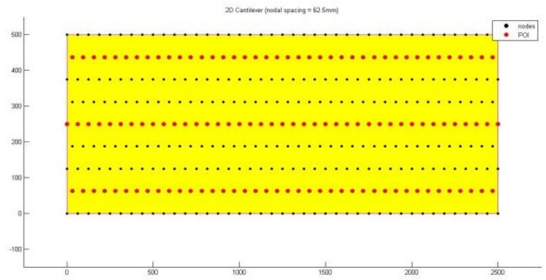

Figure 12: *Distribution of nodes on 2D cantilever with nodal spacing of 50mm*

Since the number of POI (121) is more in the domain the more exact deformation of the 2D cantilever is being captured (error = 1.5%).

Exercise 5: Nodal spacing = 50 mm (Total number of nodes = 556)

As the nodal spacing is reduced to 50mm, the solution is almost converging to the actual (error = 0.1%).

<u>Convergence Studies:</u>
The displacement at the free end for the different nodal combination is taken and it is compared with analytical solution for the convergence studies.

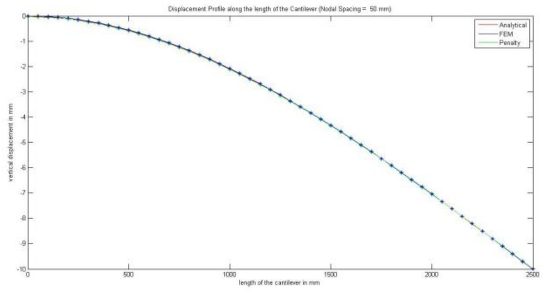

Figure 13: *Comparison of displacement profile of 2D cantilever with FEM and Analytical solutions (nodal spacing = 50mm)*

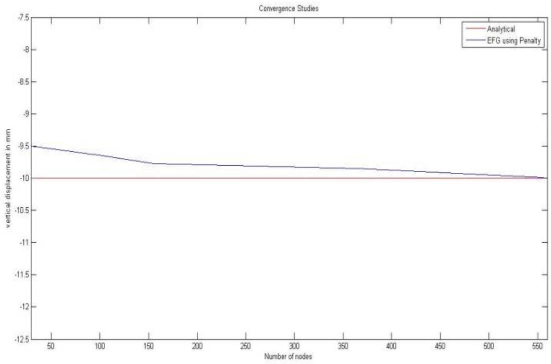

Figure 14: *Convergence study of the displacement*

It is evident from the graph that initially (number of nodes =29) the error is on the higher range (5%) and as the number of nodes increases the solution convergence and finally it matches almost with the original solution when the number of nodes increases beyond 550.
The results are tabulated along with the error percentages for the various nodal combinations.

Table 5.1: *Comparison of displacement of nodes at the free end of the 2D cantilever*

Number of nodes	**Analytical solution (mm)**	**Penalty Solution (mm)**	**Percentage of error (%)**
32	10	9.500	5
103	10	9.650	3.5
155	10	9.770	2.3
365	10	9.850	1.5
556	10	9.990	0.1

Conclusions:

- The potential applicability of the purely meshless method was scrutinized.
- Use of Background mesh is completely ruled out in evaluating stiffness matrices.
- Direct integration technique performed bypasses the complexities involved in carrying out numerical integration using Gauss quadrature technique.
- Mesh free solution converges to analytical solution with increase in number of nodes as seen from convergence study.

- The penalty method does not increase the number of unknowns and yields a positive definite matrix which is a significant advantage in practical applications
- The symmetry and the bandedness of the system matrix are preserved as long as proper penalty co-efficient is chosen.
- Computational cost required by Penalty approach is lesser than that of Lagrange multiplier method.

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